The Banks of the Cohomology River

David Eisenbud and Frank-Olaf Schreyer [*]

To the Memory of Masaki Maruyama

Abstract

We give sharp bounds on the vanishing of the cohomology of a tensor product of vector bundles on $\mathbb{P}^n$ in terms of the vanishing of the cohomology of the factors. For this purpose we introduce regularity indices generalizing the Castelnuovo-Mumford regularity.

As an application we give a sufficient condition for a vector bundle to have an unobstructed deformation theory that depends only on the cohomology table of the bundle. We construct complete families of bundles with such cohomology tables.

1 Introduction

If we know which cohomology groups of (all twists of) two vector bundles $\mathcal{F}, \mathcal{G}$ on $\mathbb{P}^n$ are zero and nonzero, what can we say about the cohomology of twists of $F \otimes G$? For example, one might naively suppose that if $H^i \mathcal{F}(a) \neq 0$ and $H^j \mathcal{G}(b) \neq 0$, and if $i + j \leq n$, then at least for some sheaves with the given vanishing pattern one might have $H^{i+j} \mathcal{F} \otimes \mathcal{G}(a + b) \neq 0$.

In this paper we will give sharp bounds on which cohomology groups of twists of $\mathcal{F} \otimes \mathcal{G}$ vanish, and we will see that they are much more restrictive than the naive idea above would suggest (see Example 1.3).

We were led to these bounds by a result from the Boij-Söderberg theory of cohomology tables of vector bundles: by Theorems 0.5 and 6.2 of our paper [6]

there is, for any vector bundle $\mathcal{F}$ on projective space a uniquely defined homogeneous vector bundle (that is, a direct sum of twists of Schur functors applied to the tangent bundle) that has, in characteristic zero, the same cohomology table as $\mathcal{F}$ up to a rational multiple. The inequalities we prove are sharp for these homogeneous bundles in characteristic zero, in a rather strong sense.

The inequalities we give strengthen those of Sidman [11] and Caviglia [2]. Those authors' work is based on the characterization of regularity in terms of "approximate" free resolutions—that is, free complexes that are resolutions away from some low-dimensional locus (the idea of using such approximate resolutions seems to go back to the paper of Gruson-Lazarsfeld-Peskine [10].) The improvement that gives us stronger bounds is the use of (approximate) free monads instead of resolutions. These ideas are described in § 3.

One of the most interesting tensor products of two bundles is $End(\mathcal{F}) = \mathcal{F}^* \otimes \mathcal{F}$, and one of its most interesting cohomology groups is

$$H^2(\mathcal{F}^* \otimes \mathcal{F}) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}),$$

the obstruction space for deformations of $\mathcal{F}$. The bounds on cohomology of a tensor product allow us to give an interesting sufficient condition under which this deformation space is zero, so that the local deformation space of $\mathcal{F}$ is smooth. It turns out that in these unobstructed cases we can actually write down complete families of the bundles with the given cohomology that are irreducible smooth rational varieties. This application occupies § 6.

The bounds will be given in terms of *regularity indices*, which we will now describe.

1.1 Regularity Indices

Let $\mathbb{K}$ be a field, and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$. We define the *k-th regularity index* of $\mathcal{F}$ to be

$$\mathrm{reg}^k \mathcal{F} := \inf\{m \mid H^j \mathcal{F}(m-j) = 0 \text{ for all } j > k\}.$$

and the k-th coregularity index to be

$$\mathrm{coreg}^k \mathcal{F} := \sup\{m \mid H^j \mathcal{F}(m-j) = 0 \text{ for all } j < n - k\}.$$

(This definition *differs* from the one in [6]!) Thus $\mathrm{reg}^0 \mathcal{F} \leq m$ if and only if $\mathcal{F}$ is m-regular in the classical sense of Castelnuovo-Mumford. For any vector bundle

$\mathcal{F}$ and any integer m we have $\operatorname{coreg}^m \mathcal{F} = -\operatorname{reg}^m \mathcal{F}^* - 1$, as one sees easily by duality.

The *cohomology table* of $\mathcal{F}$ is the collection of numbers

$$\gamma(\mathcal{F}) = \{h^i(\mathcal{F}(d)) := \dim_{\mathbb{K}} H^i(\mathcal{F}(d)\}.$$

We display it in a table with $h^i(\mathcal{F}(d))$ in the i-th row (numbering from the bottom) and the $(i+d)$-th column (numbering from left to right), and to simplify the picture we replace the zeros elements by dots. As explained in our [3] the cohomology table, in this form, is also the Betti table of the Tate resolution associated to $\mathcal{F}$, and it follows that If $\operatorname{reg}^k \mathcal{F} = m$ then (as in the case of the Castelnuovo-Mumford regularity, $k = 0$) we have $H^j \mathcal{F}(m' - j) = 0$ for all $j > k$ for every $m' > m$, and similarly for the coregularity (Proposition 3.3.)

For example, the cohomology table of the Horrocks-Mumford bundle $\mathcal{F}_{HM}$ on $\mathbb{P}^4$ is:

```
4:  100 35   4   .   .  . .    .    .    .    .
3:    .  2  10  10   5  . .    .    .    .    .
2:    .  .   .   .   .  2 .    .    .    .    .
1:    .  .   .   .   .  . 5   10   10    2    .
0:    .  .   .   .   .  . .    .    4   35  100
     -5 -4  -3  -2  -1  0 1    2    3    4    5
```

In this display, $\operatorname{reg}^m \mathcal{F}$ is the number of the left-most column with only dots above row m, and $\operatorname{coreg}^m \mathcal{F}$ is the number of the right-most column with only dots below row $n - m$. Thus for example $\operatorname{reg}^1 \mathcal{F}_{HM} = 1$ and $\operatorname{coreg}^0 \mathcal{F}_{HM} = -5$

1.2 Banks of the Cohomology River

We think of the nonzero values of the cohomology of $\mathcal{F}$ as forming the *cohomology river*, and the regularities and coregularities as defining its *banks*. Our first main result describes the banks of the cohomology river of a tensor product:

Theorem 1.1. *If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles on $\mathbb{P}^n$ then*

$$\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G}) \leq \min_{k+l=p} \left(\operatorname{reg}^k \mathcal{F} + \operatorname{reg}^l \mathcal{G} \right)$$

and

$$\operatorname{coreg}^p(\mathcal{F} \otimes \mathcal{G}) \geq 1 + \max_{k+l=p} \left(\operatorname{coreg}^k \mathcal{F} + \operatorname{coreg}^l \mathcal{G} \right).$$

The inequality for $\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G})$ holds, more generally, for any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ such that the dimension of the sheaf $Tor_1(\mathcal{F}, \mathcal{G})$ is at most $p + 2$.

3

Our second main result shows that Theorem 1.1 is sharp in a strong sense:

Theorem 1.2. *Given any pair of cohomology tables Φ, Γ of vector bundles on $\mathbb{P}^n$, there exists a pair of homogeneous vector bundles $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^n_{\mathbb{C}}$ whose cohomology tables are rational multiples of Φ and Γ, and such that equality holds for every p in the formulas for $\mathrm{reg}^p(\mathcal{F} \otimes \mathcal{G})$ and $\mathrm{coreg}^p(\mathcal{F} \otimes \mathcal{G})$ of Theorem 1.1.*

The proofs are given in § 4.

Example 1.3. In fact, the formulas of Theorem 1.1 seem to be sharp rather often. For example, let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the projection defined by the symmetric functions as in [6]. Taking $\mathcal{F} = \pi_* \mathcal{O}(4, 1, -1)$ and $\mathcal{G} = \pi_* \mathcal{O}(3, -1, -2)$ we get bundles with cohomology tables:

```
3:  70  24    .    .   .   .   .     .
2:   .   .    8    6   .   .   .     .
1:   .   .    .    .   4   .   .     .
0:   .   .    .    .   .  18  56   120
    -4  -3   -2   -1   0   1   2     3
```

and

```
3:  168  84  30    .    .   .   .   .
2:    .   .   .    .   12  12   6   .   .
1:    .   .   .    .    .   .   .   .   .
0:    .   .   .    .    .   .  12  42
     -4  -3  -2   -1    0   1   2   3.
```

Computing the cohomology table of the tensor product in Macaulay2 ([9]) we get:

```
3:  624  216   72    8    .   .   .    .
2:    .    96  140  144  96  42   .    .
1:    .     .    .    .  18  36  48    .
0:    .     .    .    .   .   .  24  216
     -4    -3   -2   -1   0   1   2    3
```

Inspection shows that equality is achieved here for all the bounds of Theorem 1.1. Notice that we have (for example) $H^1\mathcal{F}(-1) = 4$ and $H^2\mathcal{G}(-1) = 6$, but that no sheaves with these vanishing patterns can have $H^3(\mathcal{F} \otimes \mathcal{G}(-2)) \neq 0$.

Two values of the cohomology of a tensor product of bundles are particularly interesting: $H^1(\mathcal{F}^* \otimes \mathcal{F}) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ is the space of first order deformations of $\mathcal{F}$, and $H^2(\mathcal{F}^* \otimes \mathcal{F}) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ is the obstruction space.

As an application of our theory we determine, in § 6, the cohomology tables of bundles that force the obstruction space to be trivial, and we compute rational families of bundles containing all bundles with such cohomology tables.

## 2 Boij-Söderberg Theory for Vector Bundles

By a *homogeneous bundle* on $\mathbb{P}^n$ we mean the result of applying a Schur functor S_λ to the universal n-quotient bundle Q, and then (possibly) tensoring with a line bundle. Here $\lambda = \lambda_{n-1}, \ldots, \lambda_0$ is a partition with n parts; that is, the λ_i are integers such that $\lambda_{n-1} \geq \cdots \geq \lambda_0 \geq 0$. We choose our conventions so that $S_{m,0,\ldots,0}Q$ is the m-th symmetric power of Q while $S_{1^m,0,\ldots,0}Q$ is the m-th exterior power of Q. We draw the Young diagram corresponding to λ by putting λ_i boxes in the i-th row and right-justifiying the picture; for example the partition $(7, 5, 2, 2, 0, 0)$ corresponds to the diagram
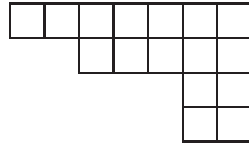
(where rows 0 and 1 have zero boxes!)

The vanishing part of Bott's Theorem about homogeneous bundles in characteristic zero has a very simple statement in terms of cohomology tables (this was pointed out to us by Jerzy Weyman).

Theorem 2.1 (Bott). *Let $\lambda_{n-1}, \ldots, \lambda_0$ be a partition as above, and let Q be the universal rank n quotient bundle on $\mathbb{P}^n_{\mathbb{C}}$. The cohomology table of $S_\lambda(Q)$ has the form*
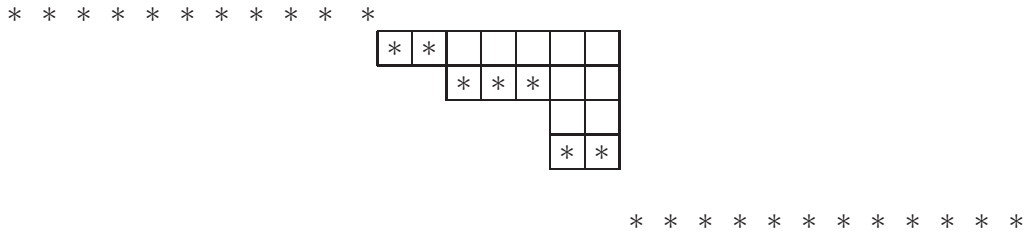
where the nonzero entries of the table are exactly those marked by ∗, *the top row of the Young diagram is row* $n - 1$, *and the right-hand column of the Young diagram is column* -1.

For example, we see from Theorem 2.1 that

$$\operatorname{reg}^k S_\lambda(Q) = -\lambda_k.$$

We partially order the partitions component-wise—in terms of Young diagrams this is the partial order by inclusion. One of the main results of Boij-Söderberg theory for vector bundles can be thought of as associating to any vector bundle on projective space a homogeneous bundle with (in characteristic zero) the same cohomology table, up to a rational multiple. We restrict ourselves to 0-regular bundles for simplicity; of course we can apply the result to any bundle by first tensoring with a sufficiently positive line bundle. The following statement combines Theorems 0.5 and 6.2 of our paper [6].

Theorem 2.2. *The cohomology table of any bundle* $\mathcal{F}$ *with* $\operatorname{reg}^0 \mathcal{F} \leq 0$ *can be written uniquely as a positive rational linear combination of the (characteristic zero) cohomology tables of a sequence of homogeneous bundles corresponding to a totally ordered set of Young diagrams.*

Remark: One can use the Boij-Söderberg decomposition to bound the numbers in the cohomology table of the tensor product using just the knowledge of which entries of the cohomology table are zero and the Hilbert polynomial. But one might hope for a still stronger principle, asserting perhaps that if the cohomology tables Φ and Γ of bundles F and G have Boij-Söderberg decompositions

$$\Phi = \sum \alpha_i \Phi^i, \quad \Gamma = \sum \beta_j \Gamma^j,$$

where Φ^i and Γ^j are the cohomology tables of the homogeneous bundles $S_{\phi^i} Q$ and $S_{\gamma^j} Q$, then the cohomology table of $F \otimes G$ would be bounded, term by term, by the sum of $\alpha_i \beta_j$ times the cohomology table of $S_{\phi^i} Q \otimes S_{\gamma^j} Q$. This is false, as Example 1.3 shows.

3 Linear Monads and Regularity Indices

It is well known that the (Castelnuovo-Mumford) regularity $\operatorname{reg}^0 \mathcal{F}$ of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ can be characterized as the smallest integer m such that $\mathcal{F}(m)$

admits a *linear free resolution*, that is, such that there is a complex

$$\mathcal{M} : \cdots \to \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\beta_{t+1}} \to \mathcal{O}_{\mathbb{P}^n}(-t)^{\beta_t} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \to 0$$

with homology $\mathrm{H}^0(\mathcal{M}) = \mathcal{F}$ and no other homology. Our next result is a characterization of this sort for all the regularity indices of a sheaf and also the coregularity indices of a vector bundle. This is the characterization that we will use to prove our main theorem.

Recall that a *monad* $\mathcal{M}$ for a sheaf $\mathcal{F}$ is a finite complex of sheaves

$$\cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \ldots$$

whose only homology is $\mathrm{H}^*(\mathcal{M}) = \mathrm{H}^0(\mathcal{M}) \cong \mathcal{F}$. The monad is called *linear* if $\mathcal{M}^i$ is a direct sum of copies of $\mathcal{O}(i)$ for each i.

Proposition 3.1. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ then $\mathrm{reg}^k \mathcal{F}$ is the smallest integer m such that $\mathcal{F}(m)$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell > k$. If $\mathcal{F}$ is a vector bundle then $\mathrm{coreg}^k \mathcal{F}$ is the largest integer m such that $\mathcal{F}(m+1)$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell < -k$.*

Proof. If $\mathcal{F}$ is a vector bundle then the dual of a monad for $\mathcal{F}$ is a monad for $\mathcal{F}^*$. Using the formula $\mathrm{coreg}^k \mathcal{F} = -\mathrm{reg}^k \mathcal{F}^* - 1$ we see that the second statement of the Proposition follows from the first.

Twisting by $-m$, the first statement will follow if we show that a coherent sheaf $\mathcal{F}$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell > k$ if and only if $\mathrm{reg}^k \mathcal{F} \leq 0$. The "only if" part follows from a standard argument in homological algebra. Here is a general version whose strength we will use later:

Lemma 3.2. *Let*

$$\mathcal{M} : \quad \cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \cdots$$

be a complex of sheaves, and let $\mathcal{F}^i := \mathrm{H}^i \mathcal{M}$ be the homology of $\mathcal{M}$ at the i-th term. If

(1) $\qquad\qquad\qquad H^{j-t}(\mathcal{M}^t) = 0 \quad$ *for all t,*

(2) $\qquad\qquad\qquad H^{j-t-1}(\mathcal{F}^t) = 0 \;$ *for all $t > 0$,*

(3) $\qquad\qquad\qquad H^{j-t+1}(\mathcal{F}^t) = 0 \;$ *for all $t < 0$*

then $H^j(\mathcal{F}^0) = 0$.

Proof. Break $\mathcal{M}$ into the short exact sequences

$$0 \to \mathcal{Z}^i \to \mathcal{M}^i \to \mathcal{B}^{i+1} \to 0$$
$$0 \to \mathcal{B}^i \to \mathcal{Z}^i \to \mathcal{F}^i \to 0$$

and chase the corresponding long exact sequences in cohomology. $\qquad\square$

To complete the proof of Proposition 3.1 we must show that if $\operatorname{reg}^k \mathcal{F} \leq 0$ then $\mathcal{F}$ admits a linear monad with $\mathcal{M}^\ell = 0$ for all $\ell > k$. The object we need is the one mentioned in [3] Example 8.5, and is constructed using the Beilinson-Gel'fand-Gel'fand correspondence (BGG). Since the property we need was not spelled out there, we review the construction and add some details.

Set $W = H^0 \mathcal{O}_{\mathbb{P}^n}(1)$, and let E be the exterior algebra $E = \Lambda W^*$. The cohomology table of a coherent sheaf $\mathcal{F}$, as we have presented it, is also the Betti table of the *Tate resolution* $\mathbb{T}(\mathcal{F})$, which is a minimal graded free exact complex over E. The terms of $\mathbb{T}(\mathcal{F})$ are

$$\mathbb{T}^e(\mathcal{F}) = \oplus_{i=0}^n \operatorname{Hom}_{\mathbb{K}}(E, H^i \mathcal{F}(e - i))$$

where $H^i \mathcal{F}(e - i)$ is considered as a vector space concentrated in degree $e - i$. We consider the elements of W^* as having degree -1, so the E-module $\omega_E = \operatorname{Hom}_{\mathbb{K}}(E, K)$ is nonzero in degrees $n + 1, n \ldots, 0$, and $\operatorname{Hom}_{\mathbb{K}}(E, H^i \mathcal{F}(e - i))$ can be nonzero only in degrees $e - i + n + 1, \ldots, e - i$.

Now suppose that $\operatorname{reg}^k \mathcal{F} \leq 0$; this means that $H^j \mathcal{F}(-j) = 0$ for $j > k$. Thus $\mathbb{T}^0 \mathcal{F}$ is generated in degrees $\geq -k + n + 1$, and it follows that the graded components of $\mathbb{T}^0 \mathcal{F}$ are all zero below degree $-k$. This implies the same vanishing for the E-submodule $P = \ker(\mathbb{T}^0 \mathcal{F} \to \mathbb{T}^1 \mathcal{F})$.

To the E-module P the BGG correspondence associates a linear free complex $L(P)$ over S:

$$L(P) : \cdots \to S \otimes P_1 \xrightarrow{\partial} S \otimes P_0 \xrightarrow{\partial} S \otimes P_{-1} \longrightarrow \cdots .$$

The differential ∂ is defined to be multiplication by the element

$$\sum_{i=0}^n x_i \otimes e_i \in S \otimes E,$$

where $\{x_i\}$ and $\{e_i\}$ are dual bases of W and W^*. Since P_j is concentrated in degree j, the module $S \otimes P_j$ is a direct sum of copies of $S(-j)$.

It follows from BGG ([3] Theorem 8.1, with $\mathbb{T}' = \mathbb{T}(\mathcal{F})^{\geq 0}$ that the sheafi-fication $\mathcal{M} := \widetilde{L(P)}$ of $L(P)$ is a monad for $\mathcal{F}$. The term $\mathcal{M}^\ell$ is equal to $\mathcal{O}_{\mathbb{P}^n}(\ell)^{\dim P - \ell}$. The observation above that $P_j = 0$ for $j < -k$ implies that $\mathcal{M}^\ell = 0$ for $\ell > k$, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The correspondence between the Tate resolution and the cohomology table allows us to generalize an important fact about Castelnuovo-Mumford regularity:

Proposition 3.3. *If* $\operatorname{reg}^k \mathcal{F} = m$ *then* $H^j \mathcal{F}(m' - j) = 0$ *for all* $j > k$ *and* $m' \geq m$. *Similarly, if* $\operatorname{coreg}^\ell \mathcal{F} = m$ *then* $H^j \mathcal{F}(m - j) = 0$ *for all* $j < n - k$ *and* $m' \leq m$.

Proof. The given conditions with $m' = m$ are simply the definitions of reg^k and coreg^k. If $H^j \mathcal{F}(m' - j) \neq 0$ for some $m' > m$, then, because the Tate resolution is a minimal complex, no term of the resolution could map into the summand $H := \operatorname{Hom}_{\mathbb{K}}(E, H^j \mathcal{F}(m' - j))$, and it follows that this module would be a submodule of one of the syzygies in the resolution. Since H is an injective module over the exterior algebra, it would actually be a summand. However, H is also a free module over the exterior algebra, so this would contradict the minimality of the Tate resolution.

Since the dual of the Tate resolution is again exact and minimal, we can apply the same argument to the dual to get the corresponding statement about coregularity. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4   Proof of Theorem 1.1

The coregularity statement of Theorem 1.1 follows from the regularity statement by dualising, so it is enough to prove the latter, in its strong form. Thus we suppose that $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $\mathbb{P}^n$ with $\dim Tor_1(\mathcal{F}, \mathcal{G}) \leq p + 2$. It suffices to show that if $p = k + \ell$ then $\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G}) \leq \operatorname{reg}^k \mathcal{F} + \operatorname{reg}^\ell \mathcal{G}$. Replacing $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}(-k)$ and $\mathcal{G}(-\ell)$ respectively, we may assume $\operatorname{reg}^k \mathcal{F} = \operatorname{reg}^\ell \mathcal{G} = 0$, and we must show that for each $j > p$ we have $H^j(\mathcal{F} \otimes \mathcal{G}(-j)) = 0$.

By Proposition 3.1, the sheaf $\mathcal{F}$ has a linear monad of the form

$$\mathcal{M} : \cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \cdots \to \mathcal{M}^k \to 0$$

where $\mathcal{M}^t$ is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^n}(t)$. Since the truncated complex

$$\mathcal{M}^+ : \quad \mathcal{M}^0 \to \cdots \to \mathcal{M}^k \to 0$$

is locally split, we have

$$\ker(\mathcal{M}^0 \to \mathcal{M}^1) \otimes \mathcal{G} = \ker(\mathcal{M}^0 \otimes \mathcal{G} \to \mathcal{M}^1 \otimes \mathcal{G})$$

and it follows that $H^0(\mathcal{M} \otimes \mathcal{G}(-j)) = \mathcal{F} \otimes \mathcal{G}(-j)$.

We now apply Lemma 3.2 to the complex $\mathcal{M} \otimes \mathcal{G}(-j)$. Since the term $\mathcal{M}^t \otimes \mathcal{G}(-j)$ is a direct sum of copies of $\mathcal{G}(t-j)$, it suffices to show

(4) $\qquad\qquad H^{j-t}\mathcal{G}(t-j) = 0$ for all $t \le k$

(5) $\qquad\qquad H^{j-t-1}(H^t(\mathcal{M} \otimes \mathcal{G}(-j))) = 0$ for all $t > 0$.

(6) $\qquad\qquad H^{j-t+1}(H^t(\mathcal{M} \otimes \mathcal{G}(-j))) = 0$ for all $t < 0$.

Since $j > p = k + \ell$ and $-t \ge -k$ we have $j - t > \ell$, and (4) holds because $\mathrm{reg}^\ell \mathcal{G} = 0$.

To prove (5) we observe that $H^t \mathcal{M} \otimes \mathcal{G}(-j) = 0$ for all $t > 0$ simply because $\mathcal{M}^+$ is locally split. It remains to prove (6). But for $t < 0$ and $j > p$ the number $j - t + 1 \ge p + 3$, so it is enough to show that $\dim H^t(\mathcal{M} \otimes \mathcal{G}(-j)) = \dim H^t(\mathcal{M} \otimes \mathcal{G}) \le p + 2$.

The local splitting of $\mathcal{M}^+$ further implies that $\mathcal{Z}^0 := \ker(\mathcal{M}^0 \to \mathcal{M}^1)$ is a vector bundle, so

$$\mathcal{M}^- : \quad \cdots \to \mathcal{M}^{-2} \to \mathcal{M}^{-1} \to \mathcal{Z}^0$$

is a locally free resolution of $\mathcal{F}$. Thus for $t < 0$

$$H^t(\mathcal{M} \otimes \mathcal{G}) = Tor_{-t}(\mathcal{F}, \mathcal{G})$$

By the rigidity of Tor (Auslander, [1]), our hypothesis that $\dim Tor_1(\mathcal{F}, \mathcal{G}) \le p + 2$ implies $\dim Tor_{-t}(\mathcal{F}, \mathcal{G}) \le p + 2$ for all $t < 0$, completing the proof.

5   Proof of Theorem 1.2

The statement for the coregularity follows from that for the regularity by duality, so we restrict ourselves to the regularity formulas.

We may shift $\mathcal{F}$ and $\mathcal{G}$ and assume without loss of generality that $\mathrm{reg}^0 \mathcal{F} = \mathrm{reg}^0 \mathcal{G} = 0$. By Theorem 2.2 we can write the cohomology table Φ of $\mathcal{F}$ as a sum of cohomology tables of homogeneous bundles in characteristic 0. Since $H^i \mathcal{F}(k) = 0$ for $i > 0$ and $k \ge 0$, these bundles must in fact have $\mathrm{reg}^0 \le 0$; that

is, they all have the form $S_\lambda Q$ for some partitions λ. Since $\text{reg}^0 \mathcal{F} = 0$ we have at least one partition λ with with $\lambda_0 = 0$ occuring. Of course similar statements hold for the cohomology table Γ of $\mathcal{G}$.

Multiplying Φ and Γ by sufficiently divisible integers, we may assume that the Boij-Söderberg decompositions have positive integral—not just rational— coefficients, so that they correspond to actual homogeneous bundles.

We will complete the proof of Theorem 1.2 by showing that if $\mathcal{F}$ is a direct sum of homogeneous bundles

$$\mathcal{F} = \bigoplus_{u=0}^{v} S_{\lambda^u}, \text{ with } \lambda^0 \leq \cdots \leq \lambda^v$$

and similarly for $\mathcal{G}$, then

$$(*) \quad \text{reg}^p(\mathcal{F} \otimes \mathcal{G}) = \min_{k+l=p} \left(\text{reg}^k \mathcal{F} + \text{reg}^\ell \mathcal{G} \right)$$

for every $0 \leq p \leq n-1$. Since the inequality $\leq$ is part of Theorem 1.1, it suffices to show that the left hand side of $(*)$ is at least as large as the right hand side.

Since the k-regularity index of $S_\lambda Q$ is $-\lambda_k$, the minimum on the right hand side of $(*)$ is achieved by the minimal partition involved in the decomposition. On the other hand, the p-th regularity index of a direct sum is the maximum of the p-th regularity indices of the components, so it suffices to prove the inequality after replacing each of $\mathcal{F}$ and $\mathcal{G}$ by a single summand, corresponding to the minimal partitions in the two decompositions; that is, we may take $\mathcal{F} = S_\lambda Q$ and $\mathcal{G} = S_\mu Q$ for some partitions λ and μ.

We now have

$$\mathcal{F} \otimes \mathcal{G} = S_\lambda Q \otimes S_\mu Q = \bigoplus S_{\nu^u} Q$$

where the set of partitions ν^u (which may occur with multiplicity) is determined by the Littlewood-Richardson formula. Since the regularity indices are the negatives of the parts of the partition, this translates into the following result in representation theory:

Proposition 5.1. *Let V be an n-dimensional vector space over a field of characteristic zero, and let $0 \leq p \leq n-1$. There is a representation $S_\nu V$ appearing in $S_\lambda V \otimes S_\mu V$, such that $\nu_p \leq \max_{k+l=p} \lambda_k + \mu_l$.*

Proof. Let

$$\lambda' = \lambda_p, \lambda_{p-1}, \ldots, \lambda_0$$
$$\mu' = \mu_p, \mu_{p-1}, \ldots, \mu_0.$$

11

be the partitions obtained by truncating λ and μ. One sees from the Littlewood-Richardson formula as described, for example, in Fulton [8], that if a representation corresponding to the partition ν' occurs in $S_{\lambda'}V \otimes S_{\mu'}V$ then the representation corresponding to the partition

$$\nu = (\lambda_{n-1} + \mu_{n-1}, \ldots, \lambda_{p+1} + \mu_{p+1}, \ \nu'_p, \ldots, \nu'_0)$$

occurs in $S_\lambda V \otimes S_\mu V$. Thus we may assume from the outset that $p = n - 1$. If we set $g := \max_{k+l=n-1} \lambda_k + \mu_l$ then the termwise sum of λ with the sequence $(\mu_0, \ldots, \mu_{n-1})$, which is the reverse of μ, is a sequence of numbers $\leq g$. We want to show that in the product $S_\lambda V \otimes S_\mu V$ there occurs a representation $S_\nu V$ such that $\nu_{n-1} \leq g$.

What we wish to prove can now be expressed as a statement about the intersection ring of the Grassmannian $\mathrm{Gr}(n, n + g)$ of n-planes in $\mathbb{C}^{n+g}$ as follows: Let

$$\mathcal{V} = (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n+g} = \mathbb{C}^{n+g})$$

be a complete flag in $\mathbb{C}^{n+g}$. We write $\Sigma_\lambda(\mathcal{V})$ for the Schubert cycle in $\mathrm{Gr}(n, n+g)$ defined by

$$\Sigma_\lambda(\mathcal{V}) = \{W \in \mathrm{Gr}(n, n + g) \mid \dim W \cap V_{g+i-\lambda_{n-i}} \geq i \text{ for } 1 \leq i \leq n - 1\},$$

and similarly for $\Sigma_\mu(\mathcal{V})$. The product of the classes $[\Sigma_\lambda(\mathcal{V})]$ and $[\Sigma_\mu(\mathcal{V})]$ in the intersection ring of $\mathrm{Gr}(n, n + g)$ is the sum (with multiplicity) of the classes of those Σ_ν such that $S_\nu V$ occurs in $S_\lambda V \otimes S_\mu V$ *and* $\nu_{n-1} \leq g$. (This is explained, and the proof sketched, in § 9 of [8]. Thus our problem is to show that the intersection product $[\Sigma_\lambda(\mathcal{V})][\Sigma_\mu(\mathcal{V})]$ is nonzero. This well-known fact can be proved as follows.

Choose another flag

$$\mathcal{V}' = V'_0 \subsetneq \cdots \subsetneq V'_{n+g}$$

in general position with respect to $\mathcal{V}$. With the evident definition of $\Sigma_\mu(\mathcal{V}')$, the product above can be computed as the class of the set-theoretic intersection

$$\Sigma_\lambda(\mathcal{V}) \cap \Sigma_\mu(\mathcal{V}').$$

(This follows, for example, from Kleiman's transversality theorem.) Thus it suffices to show that this intersection is nonempty.

Since $\mathcal{V}$ and $\mathcal{V}'$ are in generic, the subspaces $V_i \cap V'_{n+g-i+1}$ are all 1-dimensional. If e_i is a basis vector for this space, then the conditions $\lambda_i + \mu_{n-1-i} \leq g$ for $0 \leq i \leq n - 1$ imply that

$$W \in \Sigma_\lambda(\mathcal{V}) \cap \Sigma_\mu(\mathcal{V}'),$$

12

where
$$W = \overline{e_{g+1-\lambda_{n-1}}, \ldots, e_{g+n-\lambda_0}}$$
so the product of the classes of these Schubert cycles in nonzero as required. $\square$

6 Unobstructed Families of Vector Bundles

Theorem 1.1 gives a criterion for the vanishing of the obstruction space $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$. In this section we describe the deformations of these unobstructed bundles. First the criterion:

Corollary 6.1. *If $\mathcal{F}$ be a vector bundle on $\mathbb{P}^n$ with either $\mathrm{reg}^0 \mathcal{F} - \mathrm{coreg}^1 \mathcal{F} \leq 3$ or $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$, then the obstruction space $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ vanishes.*

Proof. Since $\mathrm{reg}^k \mathcal{F}^* = -\mathrm{coreg}^k \mathcal{F} - 1$ the assumption gives

$$\mathrm{reg}^1(\mathcal{F} \otimes \mathcal{F}^*) \leq \min(\mathrm{reg}^0 \mathcal{F} + \mathrm{reg}^1 \mathcal{F}^*, \mathrm{reg}^1 \mathcal{F} + \mathrm{reg}^0 \mathcal{F}^*) \leq 3 - 1 = 2$$

by Theorem 1.1. Hence $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = \mathrm{H}^2(\mathcal{F} \otimes \mathcal{F}^*) = 0$. $\square$

Since replacing $\mathcal{F}$ by $\mathcal{F}^*$ interchanges the two assumption, we will focus on the case $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$ in the following. To describe all bundles satisfying this assumption we will use Beilinson monad [3], Theorem 6.1. Given a sheaf $\mathcal{F}$ on projective space the Beilinson monad

$$\mathcal{B}: \quad \ldots \to \mathcal{B}^{-1} \to \mathcal{B}^0 \to \mathcal{B}^1 \to \ldots$$

for $\mathcal{F}$ has terms

$$\mathcal{B}^e = \oplus_j H^j(\mathcal{F}(e-j)) \otimes \Omega^{j-e}(j-e).$$

$\mathcal{B}$ is obtained by applying the functor Ω to the Tate resolution $\mathbb{T}(\mathcal{F})$, where Ω is the additive functor that sends the E-module $\omega_E(i) = \mathrm{Hom}_K(E, K(i))$ to the sheaf of twisted i-forms $\Omega^i(i)$. The identification

$$\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) = \Lambda^{i-j}W^* = \mathrm{Hom}_E(\omega_E(i), \omega_E(j))$$

provides the maps.

Theorem 6.2. *Let $\mathcal{F}$ be a vector bundle with $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$ twisted such that $\mathrm{reg}^1 \mathcal{F} = 2$. Consider $A = \mathbb{T}^0 \mathcal{F}$ and $B = \mathbb{T}^1 \mathcal{F}$. There is non-empty Zariski open subset $U \subset \mathrm{Hom}_E(A, B)$ such that the kernel*

$$\mathcal{F}_\varphi = \ker(\Omega(\varphi) : \Omega A \to \Omega B)$$

is a vector bundle with the same Chern classes and rank as $\mathcal{F}$.

13

Proof. By assumption the cohomology of $\mathcal{F}$ is non-zero in a range like the indicated one.

The Beilinson monad of $\mathcal{F}$ depends only the terms in the range indicated by a $*$, since all other terms are zero or are sent to zero. In particular $\Omega(\mathbb{T}(\mathcal{F}))$ is a two term complex

$$0 \to \Omega A \to \Omega B \to 0$$

Since it is an open condition for φ to define a monad $0 \to \Omega A \to \Omega B \to 0$ and the set of φ is non empty by the existence of $\mathcal{F}$, the result follows. $\square$

Recall from [6] that a vector bundle $\mathcal{F}$ on $\mathbb{P}^n$ has *natural cohomology* in the sense of Hartshorne and Hirschowitz if, for each $d \in \mathbb{Z}$, at most one of the groups $H^i \mathcal{F}(d)$ is non-zero. The bundles $\mathcal{F}$ is called *supernatural* if, in addition, the polynomial function $\chi(\mathcal{F}(d))$ has n distinct integral roots.

Corollary 6.3. *Let $\mathcal{F}$ be a vector bundle with* $\operatorname{reg}^1 \mathcal{F} - \operatorname{coreg}^0 \mathcal{F} \leq 3$ *normalized (by tensoring with a line bundle) so that* $\operatorname{reg}^1 \mathcal{F} = 2$, *and assume that $\mathcal{F}$ has natural cohomology. Every vector bundle with natural cohomology with the same rank, Chern classes, regularity and coregularity indices arises as*

$$\mathcal{F}_\varphi = \ker(\Omega(\varphi) : \Omega A \to \Omega B)$$

for some $\varphi \in U$. In particular these vector bundles form an irreducible unirational family.

Proof. The cohomology table of any of these bundles is determined by the Hilbert polynomial $d \mapsto \chi(\mathcal{F}(d))$, since for each twist at most two terms could be nonzero due to the narrow cohomology river, and because we have natural cohomology. So they all have the same cohomology table and all arise from some $\varphi \in U$. $\square$

Note that these bundles are not necessarily stable. For example we could have a direct sums of bundles with different slopes in these family. Thus we do not speak of a moduli space.

Corollary 6.3 does not settle the existence problem for such bundles. However, it provides a computational criterion: A bundle with desired unobstructed

natural cohomology table exists if and only if a general map $\varphi \in U$ yields such a bundle. Boij-Söderberg theory characterizes the cohomology tables that can occur, up to a rational multiple. Given an integral cohomology table γ satisfying the numerical condition of Corollary 6.3 such that some multiple of γ is the cohomology table of a bundle, we conjecture that there is a number $c_0(\gamma)$ such that $c\gamma$ is the cohomology table of a bundle if and only if $c \geq c_0(\gamma)$, as in the following example.

Example 6.4. The table γ

```
4:  56 15 .  .  .   .
3:   .  . 2  .  .   .
2:   .  .  . 1  .   .
1:   .  .  .  .  .   .
0:   .  .  .  . 8  35
    -2 -1  0  1  2   3
```

"looks like" the cohomology table of a rank 4 vector bundle on $\mathbb{P}^4$, but it is not! This is because for any two 2-forms $(\eta_1, \eta_2) \in \oplus_1^2 \Lambda^2 W^* \subset E^2$ the kernel of the wedge product $\ker(\Lambda^2 W^* \to \oplus_1^2 \Lambda^4 W^*)$ is always nonzero.

Indeed, $\eta_i \in \Lambda^2 V_i$ for some 4 dimensional subspace $V_i \subset W^*$ and the annihilator of η_i has codimension one in $\Lambda^2 V_i$. Thus the intersection $\mathrm{ann}(\eta_i) \cap \Lambda^2(V_1 \cap V_2)$ has codimension at most 1 in the 3-dimensional space $\Lambda^2(V_1 \cap V_2)$ and the intersection $\mathrm{ann}(\eta_1) \cap \mathrm{ann}(\eta_2)$ is at least 1-dimensional.

However, experiments with Macaulay2 [9] convince us that every multiple $c\gamma$ with $c \geq 2$ does occur as the cohomology table of a bundle.

References

[1] M. Auslander. Modules over unramified regular local rings. Illinois J. Math. **5** (1961), 631–647.

[2] G. Caviglia. Bounds on the Castelnuovo-Mumford regularity of tensor products. Proc. Amer. Math. Soc. **135** (2007), 1949–1957

[3] D. Eisenbud, G. Fløystad and F.-O. Schreyer. Sheaf cohomology and free resolutions over exterior algebras. Trans. Amer. Math. Soc. **355** (2003), 4397–4426.

[4] D. Eisenbud, G. Fløystad and J. Weyman. The existence of pure free resolutions. Annales de l'Inst. Fourier. To appear. arXiv:0709.1529

[5] D. Eisenbud and F.-O. Schreyer. Resultants and Chow forms via exterior syzygies. J. Amer. Math. Soc. **16** (2003), 537–579.

[6] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. J. Amer. Math. Soc. **22** (2009), 859–888.

[7] D. Eisenbud and F.-O. Schreyer. Cohomology of Coherent Sheaves and Series of Supernatural Bundles. Eur. Math. Soc. (JEMS) **12** (2010), 703–722.

[8] W. Fulton. *Young tableaux*. London Mathematical Society Student Texts 35 Cambridge University Press, Cambridge, 1997

[9] D.R. Grayson and M.E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/

[10] L. Gruson, R. Lazarsfeld, and C. Peskine. On a Theorem of Castelnuovo and the Equations defining Space Curves. Invent. Math. **72** (1983), 491–506.

[11] J. Sidman. On the Castelnuovo-Mumford regularity of products of ideal sheaves. Adv. Geom. **2** (2002), 219–229.

Author Addresses:

David Eisenbud
Department of Mathematics, University of California, Berkeley, Berkeley CA 94720
eisenbud@math.berkeley.edu

Frank-Olaf Schreyer
Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany
schreyer@math.uni-sb.de